Last time Ken explained how to write the Reidemeister torsion in terms of combinatorial Laplacians:

$$\log \tau(C) = \frac{1}{2} \sum_{q=0}^{N} (-1)^{q+1} q \log \det(\Delta_q^{(c)}),$$

where C is a finite chain complex

$$C_N \underset{\partial^*}{\overset{\partial}{\rightleftarrows}} C_{N-1} \underset{\partial^*}{\overset{\partial}{\rightleftarrows}} \cdots \underset{\partial^*}{\rightleftarrows} C_1 \underset{\partial^*}{\overset{\partial}{\rightleftarrows}} C_0$$

∂ = the boundary operator
∂^* = the adjoint (according to the inner product on C for which the basis elements are orthogonal),

$$\Delta_q^{(c)} = \partial\partial^* + \partial^*\partial : C_q \to C_q \leftarrow \text{combinatorial Laplacian}$$

- $\Delta_q^{(c)}$ = non-negative symmetric matrix encoding combinatorial information about the chain complex.

The definition of $\tau(C)$ makes sense, when C is acyclic $\iff$ all the homology groups of C are zero $\iff$ $\Delta_q^{(c)}$ does not have any zero eigenvalues.

. In 1971 Daniel Ray and Isadore Singer published "R-torsion and the Laplacian on Riemannian manifolds" in Advances of Math.

In this paper they defined an analytic analog of R-torsion in the context of closed orientable Riemannian manifolds. They showed that their analytic torsion is independent of the Riemannian metric and conjectured that in the acyclic case

A-torsion = R-torsion.

This conjecture was proved by

- Jeff Cheeger, Annals of Math, 1979
- Werner Müller, Advances of Math, 1978
- Jean-Michel Bismut and Weiping Zhang, 1992 Astérisque

Advantages of A-torsion:

- Can be generalized to complex manifolds, where it becomes a complex structure invariant (Ray-Singer)
- Can be defined for families of manifolds (Quillen)

. Formula for A-torsion:

$$\log T(M) = \frac{1}{2}\sum_{q=0}^{N}(-1)^{q+1} q \log \text{"}\det(\Delta_q)\text{"}$$

Here M is a closed orientable Riemannian manifold and Δ_q is the associated Hodge Laplacian on differential q-forms.

(Δ_q is a symmetric (self-adjoint) infinite dimensional matrix, i.e. it has countably many eigenvalues (of finite multiplicity)

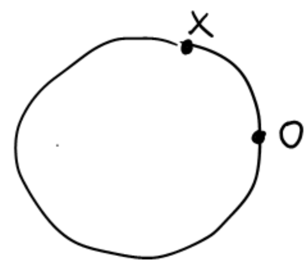
$$0 \le \lambda_0^q \le \lambda_1^q \le \dots \le \lambda_n^q \le \dots \to +\infty \;)$$

" $\det \Delta_q$ " refers to the need to make sense of this number.

Example: $M = S^1 \leftarrow$ circle of length L.

Modern Analysis (Harmonic analysis in particular) started from the following problem: describe the heat diffusion in a circular wire of length L: $u(x,t) \leftarrow$ temperature in the location x on a circle at a time t

heat equation $\rightarrow$
$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} = -\Delta u \\ u(x,0) = f(x) \end{cases}$$
$$(f(x+L) = f(x))$$



The idea for solving this equation is to find all possible basic" solutions of the heat equation, i.e. solutions in the form $u_j(x,t) = e^{-\lambda_j t}\varphi_j(x)$.

After plugging in, we see that $\varphi_n(x)$ must satisfy
$$\Delta \varphi_j = \lambda_j \varphi_j$$
and be periodic with period L.

We see that $\lambda_j = \left(\frac{2\pi n}{L}\right)^2$,

λ_0 has multiplicity 1, $\varphi_0 = \frac{1}{\sqrt{L}} \cdot 1$

λ_j's all have multiplicity 2 with corresponding eisenfunctions $\phi_j(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{2\pi n x}{L}\right)$ or $\sqrt{\frac{2}{L}}\cos\left(\frac{2\pi n x}{L}\right)$.

Then the general solution becomes

$$u(x,t) = \sum_{j=0}^{\infty} c_j e^{\lambda_j t} \phi_j(x)$$

Matching $u(x,t)$ to the initial condition, we get

$$u(x,0) = \sum_{j=0}^{\infty} c_j \phi_j(x) = f(x), \text{ so}$$

$$c_j = (f(x), \phi_j(x)) = \int_0^L f(x)\phi_j(x)\,dx$$

↑
Classical Fourier coefficients.

Putting all of this together:

$$u(x,t) = \sum_{j=0}^{\infty} \left(\int_0^L f(y)\phi_j(y)\,dy \right) \cdot e^{-\lambda_j t} \phi_j(x), \text{ so}$$

$$u(x,t) = \int_0^L K(x,y,t) f(y)\,dy,$$

where

$$K(x,y,t) = \sum_{j=0}^{\infty} e^{-\lambda_j t} \phi_j(x)\phi_j(y)$$

The heat kernel on the circle.

Fundamental solution

Formally, if $\begin{cases} \frac{\partial u}{\partial t} = -\Delta u \\ u(x,0) = f(x) \end{cases}$, then $u(x,t) = e^{-t\Delta} f(x)$,

so

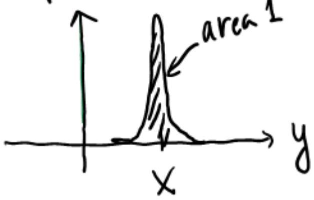
$$(e^{-t\Delta} f)(x,t) = \int_0^L K(t,x,y) f(y)\,dy$$

There is another way to construct the heat kernel.

First, observe that if $K(t,x,y)$ is the solution

of the $\begin{cases} u_t = -\Delta_x u \\ u(x,0) = \delta(x-y) \end{cases}$ K = fundamental solution,

the Dirac δ-function located at y



then $u(x,t) = \int_0^L K(t,x,y) f(y)\,dy$ will solve $\begin{cases} u_t = -\Delta u \\ u(x,0) = f(x) \end{cases}$.

This is true since K satisfies the heat equation in x variable and $\lim_{t\to 0^+} K(t,x,y) = \delta(x-y)$, $\int_0^L \delta(x-y) f(y) = f(x)$

To find the fundamental solution on the circle, we start with the fundamental solution on $\mathbb{R}$:

$$K_{\mathbb{R}}(t,x,y) = \frac{1}{\sqrt{4\pi t}} e^{-(x-y)^2/4t}.$$

Then the fundamental solution on the circle is

$$K(t,x,y) = \frac{1}{\sqrt{4\pi t}} \sum_{k\in\mathbb{Z}} e^{-\frac{|x-y-kL|^2}{4t}} \left(= \sum_{j=0}^{\infty} e^{-\lambda_j t} \phi_j(x)\phi_j(y)\right.$$

Now we will discuss how the constructions above will help us to make sense of the determinant of $-\Delta$ on the circle.

In the basis of eigenfunctions $\{\phi_j(x)\}_{j=0}^{\infty}$, the Laplacian Δ is a diagonal matrix